Week 1 Revision

- Dot product (can be defined in $\mathbb{R}^N$ for any $N \geq 1$)
- Cross product (Only defined in $\mathbb{R}^3$)
- Planes in $\mathbb{R}^3$

Dot product

projection

angle between two vectors

$\vec{u} \cdot \vec{v} = 0$ iff $\vec{u} \perp \vec{v}$

Measuring the difference between the directions of two vectors

$$\left| \frac{\vec{u}}{|\vec{u}|} - \frac{\vec{v}}{|\vec{v}|} \right|^2 = 2 - 2\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|} \right)$$

- If θ is the included angle between $\vec{u}, \vec{v}$,

then
$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\cdot|\vec{v}|} \qquad (0 \leq \theta \leq \pi)$$

(proved by cosine formula)

This holds for $\mathbb{R}^2$ and $\mathbb{R}^3$

- Projection :

Projection of $\vec{u}$ on $\vec{v}$ $= \dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v}$

- This makes sense because the length of $\vec{v}$ shall have no effect for the projection vector.

Application :

- a way to generate a set of orthogonal (perpendicular) vectors.
- Measuring the distance between a point and a line (or a point and a plane)

(Lecture)

$\mathbb{R}^2$: distance between a point and a line

$\mathbb{R}^3$: (i) distance between a point and a line

(ii) distance between a point and a plane.

(Assignment 1)

Example :

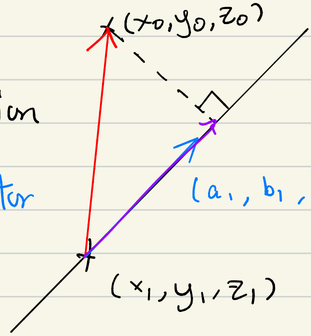
In $\mathbb{R}^3$, let

$$l_1 : \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$$ be a straight line

and (x_0, y_0, z_0) be a point.

Find the formula of the distance.

Note that (x_1, y_1, z_1) is a point on the line and (a_1, b_1, c_1) is a direction of the line.



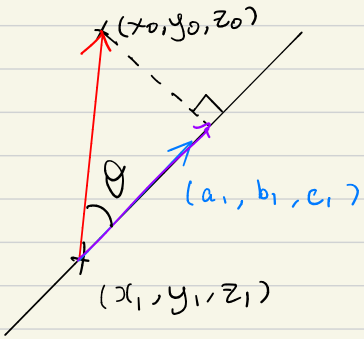
Purple vector is the projection of red vector on blue vector

Purple vector

$$= \frac{(x_0 - x_1, y_0 - y_1, z_0 - z_1) \cdot (a_1, b_1, c_1)}{a_1^2 + b_1^2 + c_1^2} (a_1, b_1, c_1)$$

Distance

$$= \left| (x_0 - x_1, y_0 - y_1, z_0 - z_1) - \frac{(x_0 - x_1, y_0 - y_1, z_0 - z_1) \cdot (a_1, b_1, c_1)}{a_1^2 + b_1^2 + c_1^2} (a_1, b_1, c_1) \right|$$



A neater formula for the distance is

$$\frac{|(x_0 - x_1, y_0 - y_1, z_0 - z_1) \times (a_1, b_1, c_1)|}{|(a_1, b_1, c_1)|}$$

Another characterization of the projection :

Projection of $\vec{u}$ on $\vec{v}$ is a vector $\vec{w}$ parallel to $\vec{v}$, so that

$$(\vec{u}-\vec{w}) \perp \vec{v} :$$

If we put $\vec{w} = c\vec{v}$ for some $c \in \mathbb{R}$

When we require $(\vec{u}-\vec{w}) \perp \vec{v}$, i.e.

$$(\vec{u}-\vec{w}) \cdot \vec{v} = 0$$

we must have $\vec{u} \cdot \vec{v} - c|\vec{v}|^2 = 0$

$$c = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}$$

It forces $\vec{w} = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v}$ #

Note that $\vec{u} - \vec{w} = \vec{u} - \dfrac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v}$

$\{\vec{u}-\vec{w}, \vec{v}\}$ is a set of orthogonal vectors, so that it has the same span as $\{\vec{u}, \vec{v}\}$

Example : [This holds for $\mathbb{R}^N$, any $N \geq 1$]

Given vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ s.t the set is linearly independent, we can derive a set of orthogonal vectors $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ with the same span.

Let $\vec{w}_1 = \vec{v}_1$

$$\vec{w}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{w}_1}{|\vec{w}_1|^2}\vec{w}_1 \qquad (\vec{w}_2 \perp \vec{w}_1)$$

$$\vec{w}_3 = \vec{v}_3 - \frac{\vec{v}_3 \cdot \vec{w}_2}{|w_2|^2}\vec{w}_2 - \frac{\vec{v}_3 \cdot \vec{w}_1}{|\vec{w}_1|^2}\vec{w}_1$$

$$\begin{pmatrix} \vec{w}_3 \perp \vec{w}_2 \\ \vec{w}_3 \perp \vec{w}_1 \end{pmatrix}$$

Cross product.

- Cross product is only defined in $\mathbb{R}^3$
- Let $\vec{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k} \in \mathbb{R}^3$

$\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k} \in \mathbb{R}^3$

$$\vec{u} \times \vec{v} \overset{\text{def}}{=} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

It turns out that $\vec{u} \times \vec{v}$ is a vector

(i) perpendicular to both $\vec{u}, \vec{v}$

(following right hand rule)

(ii) with magnitude $= |\vec{u}||\vec{v}||\sin\theta|$

where θ is the included angle between vectors $\vec{u}, \vec{v}$

This one is verified in assignment 1, Q2(ii)

(Rmk: So do not attempt this question by assuming $|\vec{u}\times\vec{v}| = |\vec{u}|\cdot|\vec{v}||\sin\theta|$)

- $\vec{u} \times \vec{v} = \vec{0}$ iff $\vec{u}, \vec{v}$ are parallel
- If $\vec{u}, \vec{v} \in \mathbb{R}^3$ are linearly independent,

  then $\{\vec{u}, \vec{v}, \vec{u} \times \vec{v}\}$ are linearly independent.

(In particular, this means that

$\{\vec{u}, \vec{v}, \vec{u} \times \vec{v}\}$ is a basis of $\mathbb{R}^3$)

Proof: Let $c_1, c_2, c_3 \in \mathbb{R}$.

Suppose $c_1\vec{u} + c_2\vec{v} + c_3\vec{u} \times \vec{v} = 0$ — ①

Since $\vec{u} \cdot (\vec{u} \times \vec{v}) = \vec{0} = \vec{v} \cdot (\vec{u} \times \vec{v})$,

Taking dot product with $\vec{u} \times \vec{v}$ on both sides of

①, we have

$$\vec{0} + c_3 |\vec{u} \times \vec{v}|^2 = 0$$

But $\vec{u}, \vec{v}$ are not parallel $\Rightarrow$ $|\vec{u} \times \vec{v}| \neq 0$

This forces $c_3 = 0$

① becomes $c_1\vec{u} + c_2\vec{v} = 0$

By linear independence of $\vec{u}, \vec{v}$, we have

$c_1 = c_2 = 0$ ※

- Area of $\Delta ABC = \frac{1}{2}|\vec{AB} \times \vec{AC}|$

Planes and straight lines in $\mathbb{R}^3$:

- Given simultaneous eq$^{\underline{n}}$s

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} \qquad (a_1, b_1, c_1 \neq 0)$$

It represents a straight line in $\mathbb{R}^3$

with direction $= (a_1, b_1, c_1)$

and passing through (x_1, y_1, z_1)

- Given an equation

$ax + by + cz + d = 0$ with $(a, b, c) \neq (0, 0, 0)$

It represents a plane with

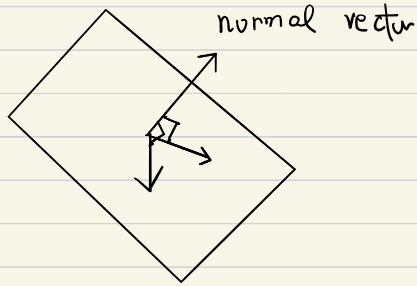
direction $= (a, b, c)$ and

passing through the point

$$\begin{cases} \left(-\frac{d}{a}, 0, 0\right) & \text{if } a \neq 0 \\ \left(0, -\frac{d}{b}, 0\right) & \text{if } b \neq 0 \\ \left(0, 0, -\frac{d}{c}\right) & \text{if } c \neq 0 \end{cases}$$

- Direction of a plane here means a normal vector

(a vector which is perpendicular to the plane)



A plane can be determined by a normal vector (direction) and one point the plane contains.

- Suppose $\vec{u}, \vec{v}$ are linearly independent vectors parallel to a plane Π.

  Let (x_0, y_0, z_0) be a pt of the plane Π.

Then, $(x, y, z) \in \Pi$

iff $\quad (x-x_0, y-y_0, z-z_0) = m\vec{u} + n\vec{v}$

for some $m, n \in \mathbb{R}$

This is equivalent to say

$$(x-x_0, y-y_0, z-z_0)\cdot(\vec{u}\times\vec{v}) = 0$$

because $\{\vec{u}, \vec{v}, \vec{u}\times\vec{v}\}$ is a basis of $\mathbb{R}^3$

why this would be the reason? check!

Therefore, the equation of plane Π is

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0$$

for $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$

- To check if A, B, C, D are coplanar:

Take a reference point, say A.

A, B, C, D are coplanar iff $\vec{AB}\cdot(\vec{AC}\times\vec{AD}) = 0$

Rmk: if A, B, C, D are not coplanar, then $|\vec{AB}\cdot(\vec{AC}\times\vec{AD})|$ will be the volume of parallelopiped with vertices A, B, C, D.

Example: Consider $A = (a_1, b_1)$, $B = (a_2, b_2)$, $C = (a_3, b_3)$ in $\mathbb{R}^2$. Then,

the area of $\Delta ABC = \frac{1}{2} \left| \det \begin{pmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{pmatrix} \right|$

absolute value of a real no.

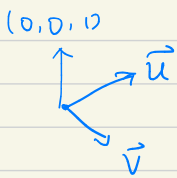
<u>Proof</u>:

$\overrightarrow{AB} = (a_2 - a_1, b_2 - b_1)$

$\overrightarrow{AC} = (a_3 - a_1, b_3 - b_1)$

let $\vec{u} = (a_2 - a_1, b_2 - b_1, 0)$

$\vec{v} = (a_3 - a_1, b_3 - b_1, 0)$



Volume of the parallelopiped $= (0,0,1) \cdot (\vec{u} \times \vec{v})$

$$= \left| \det \begin{pmatrix} 0 & 0 & 1 \\ a_2 - a_1 & b_2 - b_1 & 0 \\ a_3 - a_1 & b_3 - b_1 & 0 \end{pmatrix} \right|$$

$C_1 + a_1 C_3 \rightarrow C_2 + b_1 C_3 \rightarrow R_2 + R_1 \rightarrow R_3 + R_1$

$$= \left| \det \begin{pmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{pmatrix} \right|$$

Also, note that

Area of $\Delta ABC = \frac{1}{2}$ volume of //opiped

($\because$ altitude $= 1$)